Exercise No. 9: The Quantized Electromagnetic Field

1. Find the following commutators:
   (a) $[N_{\mathbf{k}\alpha}, \mathbf{A}]$,
   (b) $[N_{\mathbf{k}\alpha}, \mathbf{E}]$,
   (c) $[N_{\mathbf{k}\alpha}, \mathbf{B}]$,

   where $N_{\mathbf{k}\alpha}$ is the number of photons with momentum $\mathbf{k}$ and polarization α. What do these results mean?

2. (a) The angular momentum density of the electromagnetic field is $\mathbf{r} \times \mathbf{g}$ where $\mathbf{g}$ is the Poynting vector. Show that classically the total angular momentum of the field is

   $$\mathbf{J} = \frac{1}{4\pi} \int d^3r E_i (\mathbf{r} \times \nabla) A_i + \frac{1}{4\pi} \int d^3r \mathbf{E} \times \mathbf{A} \ .$$

   (b) Note that the second term has no explicit $\mathbf{r}$-dependence (it has no "arm") and can be interpreted as the intrinsic angular momentum of the field (its spin). Denote the last term by $\mathbf{S}$. Show that

   $$\mathbf{S} = \sum_{\mathbf{k}\alpha} \alpha \hat{\mathbf{k}} (c^\dagger_{\mathbf{k}\alpha} c_{\mathbf{k}\alpha} + \frac{1}{2}[c_{\mathbf{k}\alpha}, c^\dagger_{\mathbf{k}\alpha}])$$

   in the vacuum where $\alpha = \pm$ denotes the circular polarization states $\hat{\mathbf{e}}_{\mathbf{k}\pm} = \mp \frac{i}{\sqrt{2}} (\hat{\mathbf{e}}_{\mathbf{k}1} \pm i \hat{\mathbf{e}}_{\mathbf{k}2})$.

   (c) Demonstrate that the only allowable projections of the spin along the photon's momentum are $\lambda = \pm 1$. λ is called the *helicity* of the photon.

3. Calculate the commutator $[E_i(\mathbf{r}), B_j(\mathbf{r}')]$.

4. Substitute the expression for the quantized electromagnetic field in the Hamiltonian and show that the Hamiltonian then takes the form

   $$H = \int \frac{d^3k}{(2\pi)^3} \omega_{\mathbf{k}} \sum_\alpha \left(c^\dagger_{\mathbf{k}\alpha} c_{\mathbf{k}\alpha} + \frac{1}{2}[c_{\mathbf{k}\alpha}, c^\dagger_{\mathbf{k}\alpha}] \right) \ .$$