Exercise No. 1

1. Compute explicitly using the commutation relations the following commutators for the bosonic case:
   (a) $[N_{tot}, (a_i^\dagger)^n]$
   (b) $[N_{tot}, a_i^n]$
   (c) $[N_{tot}, A_{ij}^\dagger]$
   (d) $[A_{ij}^\dagger, A_{kl}^\dagger]$
   (e) $[A_{ij}, A_{kl}^\dagger]$

   where $A_{ij}^\dagger = a_i^\dagger a_j^\dagger$ and $N_{tot} = \sum_i a_i^\dagger a_i$

2. Show that for bosons:
   (a) $[a, f(a^\dagger)] = \frac{\partial f(a^\dagger)}{\partial a^\dagger}$ and $[a^\dagger, f(a)] = -\frac{\partial f(a)}{\partial a}$
   (b) $e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \ldots$

3. Let $h(x, p)$ be the Hamiltonian of a harmonic oscillator whose frequency is ω. A system is composed of two identical particles and thus its Hamiltonian is $H = h(x_1, p_1) + h(x_2, p_2)$. Find the possible state vectors, energies and degeneracies of the ground state and the first excited state for
   (a) two electrons
   (b) two spin 0 bosons.

4. An asymmetric beam-splitter (figure on the left) is a beam-splitter in which the transmission and reflection are given by $|1\rangle \to \alpha|2\rangle + \beta|3\rangle$, where $|\alpha|^2 + |\beta|^2 = 1$ (there are additional constraints on α and β). Find the probabilities of the various outcomes of the following experiment with an asymmetric beam-splitter whose initial state is two photons at $|1\rangle$ and $|2\rangle$ (figure on the right).