Exercise No. 3: Second Quantization

1. Calculate the following commutators for fermions:
   (a) $[N_{tot}, (a_i^\dagger)^n]$
   (b) $[N_{tot}, a_i^n]$
   (c) $[N_{tot}, A_{ij}^\dagger]$
   (d) $[A_{ij}, A_{kl}^\dagger]$
   (e) $[A_{ij}, A_{kl}]$
   where $A_{ij}^\dagger = a_i^\dagger a_j^\dagger$ and $N_{tot} = \sum_i N_i = \sum_i a_i^\dagger a_i$.

2. Show that for fermions the particle number operator $N = \sum_i a_i^\dagger a_i$ commutes with the Hamiltonian
   \[ H = \sum_{ij} \langle i|T|j\rangle a_i^\dagger a_j + \frac{1}{2} \sum_{ijkl} \langle ij|V|kl\rangle a_i^\dagger a_j^\dagger a_l a_k. \]

3. Consider the Hamiltonian
   \[ H = \epsilon a_i^\dagger a_i + \sum_k \epsilon_k a_k^\dagger a_k + a_i^\dagger a \sum_k M_k (a_k + a_k^\dagger), \]
   where $\epsilon$ is the fermion energy of a particular state, $\epsilon_k$ is the energy of a phonon (which is a boson) with a wave vector $k$ and $M_k$ are $k$-dependent coupling constants. Define the following transformation for any operator $A$:
   \[ \tilde{A} = e^S A e^{-S} = A + [S, A] + \frac{1}{2!} [S, [S, A]] + \ldots \]
   where $S$ is an operator. In particular we take $S = a^\dagger a \sum_k \frac{M_k}{\epsilon_k} (a_k^\dagger - a_k)$.
   (a) Explain the possible physical meaning of every term in the Hamiltonian.
   (b) Show that $\tilde{a} = aX$, $\tilde{a}^\dagger = a^\dagger X^\dagger$, $\tilde{a}_k = a_k - \frac{M_k}{\epsilon_k} a^\dagger a$, $\tilde{a}_k^\dagger = a_k^\dagger - \frac{M_k}{\epsilon_k} a^\dagger a$
      where $X = \exp \left[ - \sum_k \frac{M_k}{\epsilon_k} (a_k^\dagger - a_k) \right]$ (note that $X^\dagger = X^{-1}$).
   (c) Show that $\tilde{H} = a^\dagger a (\epsilon - \Delta) + \sum_k \epsilon_k a_k^\dagger a_k$ where $\Delta = \sum_k \frac{M_k^2}{\epsilon_k}$. Verify that this form agrees with your initial interpretation of $H$. 

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