Exercise No. 3: Second Quantization

1. Calculate the following commutators for fermions:
   (a) $[N_{tot}, (a_i^\dagger)^n]$
   (b) $[N_{tot}, a_i^n]$
   (c) $[N_{tot}, A_{ij}^\dagger]$
   (d) $[A_{ij}^\dagger, A_{kl}^\dagger]$
   (e) $[A_{ij}, A_{kl}^\dagger]$

   where $A_{ij}^\dagger = a_i^\dagger a_j^\dagger$ and $N_{tot} = \sum_i N_i = \sum_i a_i^\dagger a_i$.

2. Show that for fermions the particle number operator $N = \sum_i a_i^\dagger a_i$ commutes with the Hamiltonian

$$H = \sum_{ij} \langle i|T|j\rangle a_i^\dagger a_j + \frac{1}{2} \sum_{ijkl} \langle ij|V|kl\rangle a_i^\dagger a_j^\dagger a_l a_k.$$

3. Consider the Hamiltonian

$$H = \epsilon a^\dagger a + \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + a^\dagger a \sum_{\mathbf{k}} M_{\mathbf{k}} (a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger),$$

   where ϵ is the fermion energy of a particular state, $\epsilon_{\mathbf{k}}$ is the energy of a phonon (which is a boson) with a wave vector $\mathbf{k}$ and $M_{\mathbf{k}}$ are $\mathbf{k}$-dependent coupling constants. Define the following transformation for any operator A:

$$\bar{A} = e^S A e^{-S} = A + [S, A] + \frac{1}{2!} [S, [S, A]] + \ldots$$

   where S is an operator. In particular we take $S = a^\dagger a \sum_{\mathbf{k}} \frac{M_{\mathbf{k}}}{\epsilon_{\mathbf{k}}} (a_{\mathbf{k}}^\dagger - a_{\mathbf{k}})$.

   (a) Explain the possible physical meaning of every term in the Hamiltonian.
   (b) Show that $\bar{a} = aX$, $\bar{a}^\dagger = a^\dagger X^\dagger$, $\bar{a}_{\mathbf{k}} = a_{\mathbf{k}} - \frac{M_{\mathbf{k}}}{\epsilon_{\mathbf{k}}} a^\dagger a$, $\bar{a}_{\mathbf{k}}^\dagger = a_{\mathbf{k}}^\dagger - \frac{M_{\mathbf{k}}}{\epsilon_{\mathbf{k}}} a^\dagger a$ where $X = \exp\left[-\sum_{\mathbf{k}} \frac{M_{\mathbf{k}}}{\epsilon_{\mathbf{k}}} (a_{\mathbf{k}}^\dagger - a_{\mathbf{k}}) \right]$ (note that $X^\dagger = X^{-1}$).
   (c) Show that $\bar{H} = a^\dagger a(\epsilon - \Delta) + \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}}$ where $\Delta \equiv \sum_{\mathbf{k}} \frac{M_{\mathbf{k}}^2}{\epsilon_{\mathbf{k}}}$. Verify that this form agrees with your initial interpretation of H.