Solution for the Quantum Physics 1 Exam from Jan. 23, 2003

1.

$$\rho = \frac{N}{L^2} = 2\frac{\pi p_F^2}{h^2} = \frac{2\pi \cdot 2m(\epsilon_F - V)}{h^2} = \frac{4\pi m}{h^2}\left(\epsilon_F - \frac{kx^2}{2} - ky^2\right)$$

$$N = \int_{x^2+2y^2 \le 2\epsilon_F/k} \rho dxdy = \int_{x^2+2y^2 \le 2\epsilon_F/k} \rho(x,u)dx\frac{du}{\sqrt{2}} \; (u \equiv \sqrt{2}y, r \equiv \sqrt{x^2+y^2})$$

$$\begin{aligned} N &= \frac{4\pi m}{\sqrt{2}h^2}\int_0^{\sqrt{2\epsilon_F/k}}\left(\epsilon_F - \frac{k}{2}r^2\right)2\pi r dr = \frac{8\pi^2 m}{\sqrt{2}h^2}\left[\epsilon_F\frac{r^2}{2} - \frac{k}{8}r^4\right]_0^{\sqrt{2\epsilon_F/k}} = \\ &= \frac{8\pi^2 m\epsilon_F^2}{\sqrt{2}h^2 k}\left(1 - \frac{1}{2}\right) = \frac{m\epsilon_F^2}{\sqrt{2}\hbar^2 k} \Rightarrow \epsilon_F = 2^{1/4}\hbar\sqrt{\frac{kN}{m}} \end{aligned}$$

2.

$$\Delta E = \left\langle \frac{\mathbf{p}^4}{8m^3c^2}\right\rangle, \; p_x = i\sqrt{\frac{m\hbar}{2}\sqrt{\frac{k}{m}}}(a_x^\dagger - a_x), \; p_y = i\sqrt{\frac{m\hbar}{2}\sqrt{\frac{2k}{m}}}(a_y^\dagger - a_y)$$

$$\Delta E = \frac{\hbar^2 k}{32m^2c^2}\left\langle\left[(a_x^\dagger - a_x)^4 + 2(a_y^\dagger - a_y)^4 + 2\sqrt{2}(a_x^\dagger - a_x)^2(a_y^\dagger - a_y)^2\right]\right\rangle \equiv \frac{\hbar^2 k}{32m^2c^2}X$$

The non-zero contributions to the ground state:

$$\begin{aligned} X &= \langle 0,0|(a_x a_x^\dagger a_x a_x^\dagger + a_x a_x a_x^\dagger a_x^\dagger + 2a_y a_y^\dagger a_y a_y^\dagger + 2a_y a_y a_y^\dagger a_y^\dagger + 2\sqrt{2}a_x a_x^\dagger a_y a_y^\dagger)|0,0\rangle = \\ &= 1 + 2 + 2 + 2\cdot 2 + 2\sqrt{2} = 9 + 2\sqrt{2} \end{aligned}$$

So $\Delta E_{0,0} = \frac{\hbar^2 k}{32m^2c^2}(9 + 2\sqrt{2})$. And for the first excited state:

$$\begin{aligned} X &= \langle 1,0|(a_x a_x^\dagger a_x a_x^\dagger + a_x^\dagger a_x a_x^\dagger a_x + a_x a_x a_x^\dagger a_x^\dagger + a_x a_x^\dagger a_x^\dagger a_x + a_x^\dagger a_x a_x a_x^\dagger + 2a_y a_y^\dagger a_y a_y^\dagger + \\ &\quad +2a_y a_y a_y^\dagger a_y^\dagger + 2\sqrt{2}a_x a_x^\dagger a_y a_y^\dagger + 2\sqrt{2}a_x^\dagger a_x a_y a_y^\dagger)|1,0\rangle = \\ &= 4 + 1 + 3\cdot 2 + 2 + 2 + 2 + 2\cdot 2 + 2\sqrt{2}\cdot 2 + 2\sqrt{2} = 21 + 6\sqrt{2} \end{aligned}$$

so $\Delta E_{1,0} = \frac{\hbar^2 k}{32m^2c^2}(21 + 6\sqrt{2})$.

3. *Note* The plane-wave wave-function cannot be normalized to 1 on the entire z-axis. Thus we may normalize it to give the particle-density at every point.

$$\psi(z \le 0) = a\begin{pmatrix} 1 \\ 0 \\ \frac{pc}{E+mc^2} \\ 0 \end{pmatrix} e^{ipz/\hbar} + b\begin{pmatrix} 1 \\ 0 \\ -\frac{pc}{E+mc^2} \\ 0 \end{pmatrix} e^{-ipz/\hbar}, \; \psi(z \ge 0) = f\begin{pmatrix} 1 \\ 0 \\ \frac{p'c}{E-V_0+mc^2} \\ 0 \end{pmatrix} e^{ip'z/\hbar}$$

$$p = \frac{1}{c}\sqrt{E^2 - m^2c^4} = \frac{\sqrt{5}}{2}mc, \; p' = \frac{1}{c}\sqrt{(E-V_0)^2 - m^2c^4}$$

Boundary condition at $z=0$:

$$a+b=f, \frac{pc}{E+mc^2}(a-b)=\frac{p'c}{E-V_0+mc^2}f$$

$V_0=\frac{1}{2}mc^2 \Rightarrow p'=0$

$$a+b=f, a-b=0 \Rightarrow f=2a, \rho(z)=\psi^\dagger\psi=4|a|^2$$

4. $V_0=2mc^2 \Rightarrow p'=i\frac{\sqrt{3}}{2}mc$

$$a+b=f, \frac{1}{\sqrt{5}}(a-b)=\frac{i\sqrt{3}/2}{3/2-2+1}f=i\sqrt{3}f \Rightarrow a-b=i\sqrt{15}f \Rightarrow f=\frac{2a}{1+i\sqrt{15}}$$

$$\psi=f\begin{pmatrix}1\\0\\i\sqrt{3}\\0\end{pmatrix}e^{i(i\sqrt{3}/2)mcz/\hbar} \Rightarrow \psi^\dagger\psi=(1+3)|f|^2e^{-\sqrt{3}mcz/\hbar}=|a|^2e^{-\sqrt{3}mcz/\hbar}$$

5. $V_0\to\infty \Rightarrow p'\to\infty$ (p' pure real):

$$a+b=f, \frac{1}{\sqrt{5}}(a-b)=-f \Rightarrow f=\frac{2a}{1-\sqrt{5}}$$

$$\psi=f\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}e^{ip'z/\hbar} \Rightarrow \psi^\dagger\psi=2|f|^2=\frac{2\cdot 4|a|^2}{(1-\sqrt{5})^2}=\frac{8|a|^2}{(1-\sqrt{5})^2}$$

6. The same mathematical manipulations as done in the lecture (or in Berestetskii, Lifshitz and Pitaevskii's Quantum Electrodynamics) but with $\mathbf{p}$ replaced by $\mathbf{p}-\frac{e}{c}\mathbf{A}$ lead to

$$\hat{O}=\frac{1}{8m^2c^2}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)^2.$$

7.

$$|\psi\rangle=\frac{1}{\sqrt{2}}(a_B^\dagger+ia_A^\dagger)e^{-1/2}\sum_{n=0}^{\infty}\frac{(1/\sqrt{2})^n}{\sqrt{n!}}\frac{1}{\sqrt{n!}}(a_A^\dagger+ia_B^\dagger)^n|0\rangle$$

$$\begin{aligned}P(1_A,k_B) &= |\langle 1_A,k_B|\psi\rangle|^2=\left|\frac{i}{\sqrt{2}}e^{-1/2}\left(\frac{i}{\sqrt{2}}\right)^k\frac{1}{\sqrt{k!}}+\frac{1}{\sqrt{2}}e^{-1/2}\frac{1}{i}\left(\frac{i}{\sqrt{2}}\right)^k\frac{k}{\sqrt{k!}}\right|^2= \\ &= e^{-1}\left(\frac{1}{2}\right)^{k+1}\frac{1}{k!}(1-k)^2\end{aligned}$$

$$\begin{aligned}P(1_A) &= \sum_{k_B=0}^{\infty}P(1_A,k_B)=e^{-1}\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^{k+1}\frac{1}{k!}(1-k)^2=e^{-1}\sum_{k=0}^{\infty}\frac{1}{2^{k+1}k!}\left[k(k-1)-k+1\right]= \\ &= \frac{1}{2}e^{-1}\left(\sum_{k=0}^{\infty}\frac{1}{2^k k!}-\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{2^{k-1}(k-1)!}+\frac{1}{4}\sum_{k=2}^{\infty}\frac{1}{2^{k-2}(k-2)!}\right)= \\ &= e^{-1}e^{1/2}\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{4}\right)=\frac{3}{8}e^{-1/2}.\end{aligned}$$

And from the expression for $P(1_A, k_B)$ it is clear that $P(1,1) = 0$, thus if it is given that there is one photon at B, there cannot be a single photon at A.

8. The two final states at A and at B are uncorrelated coherent states

$$|\alpha = \frac{1+i}{\sqrt{2}}\rangle = e^{-1/2} \sum_{n=0}^{\infty} \left(\frac{1+i}{\sqrt{2}}\right)^n |n\rangle,$$

so $P(1) = \left|\langle 1|\alpha = \frac{1+i}{\sqrt{2}}\rangle\right|^2 = e^{-1}\left|\frac{1+i}{\sqrt{2}}\right|^2 = e^{-1}$.

9. For α_k and $\alpha_k^\dagger$ to be annihilation and creation operators, respectively, we require $[\alpha_k, \alpha_{k'}] = 0$, $[\alpha_k^\dagger, \alpha_{k'}^\dagger] = 0$ and $[\alpha_k, \alpha_{k'}^\dagger] = \delta_{kk'}$.

$[\alpha_k, \alpha_{k'}] = -u_k v_{k'}[a_k, a_{-k'}^\dagger] - v_k u_{k'}[a_{-k}^\dagger, a_{k'}] = (-u_k v_{-k} + v_k u_{-k})\delta_{k,-k'} = 0 \Rightarrow u_k = u_{-k}, v_k = v_{-k}$

$$[\alpha_k, \alpha_{k'}^\dagger] = (u_k u_{k'} - v_k v_{k'})\delta_{kk'} = \delta_{kk'} \Rightarrow u_k^2 - v_k^2 = 1.$$

There is no vacuum for α_k since $\alpha_k|\psi\rangle = u_k a_k|\psi\rangle - v_k a_{-k}^\dagger|\psi\rangle$ and for bosons $a_{-k}^\dagger|\psi\rangle$ never cancels.