Solution for the Quantum Physics 1 Exam from Sept. 24, 2003

1. Using the beam-splitter evolution rules $a^\dagger_{1\sigma} \to \frac{1}{\sqrt{2}}\left(a^\dagger_{4\sigma} + i a^\dagger_{3\sigma}\right)$ and $a^\dagger_{2\sigma} \to \frac{1}{\sqrt{2}}\left(a^\dagger_{3\sigma} + i a^\dagger_{4\sigma}\right)$, where $a^\dagger_{i\sigma}$ is the creation operator creating a particle with polarization/spin σ at position i, we obtain for identical particles with spins σ and s

$$a^\dagger_{1\sigma}a^\dagger_{1s} \to \frac{1}{2}\left(i a^\dagger_{4\sigma}a^\dagger_{3s} + i a^\dagger_{3\sigma}a^\dagger_{4s} + a^\dagger_{4\sigma}a^\dagger_{4s} - a^\dagger_{3\sigma}a^\dagger_{3s}\right) \quad (1)$$

$$a^\dagger_{1\sigma}a^\dagger_{2s} \to \frac{1}{2}\left(a^\dagger_{4\sigma}a^\dagger_{3s} - a^\dagger_{3\sigma}a^\dagger_{4s} + i a^\dagger_{4\sigma}a^\dagger_{4s} + i a^\dagger_{3\sigma}a^\dagger_{3s}\right) \quad (2)$$

$$a^\dagger_{2\sigma}a^\dagger_{1s} \to \frac{1}{2}\left(a^\dagger_{3\sigma}a^\dagger_{4s} - a^\dagger_{4\sigma}a^\dagger_{3s} + i a^\dagger_{4\sigma}a^\dagger_{4s} + i a^\dagger_{3\sigma}a^\dagger_{3s}\right) \quad (3)$$

$$a^\dagger_{2\sigma}a^\dagger_{2s} \to \frac{1}{2}\left(i a^\dagger_{3\sigma}a^\dagger_{4s} + i a^\dagger_{4\sigma}a^\dagger_{3s} + a^\dagger_{3\sigma}a^\dagger_{3s} - a^\dagger_{4\sigma}a^\dagger_{4s}\right) . \quad (4)$$

All these transformations contain both terms of the required form and terms in which the two particles appear in the same detector. The latter states may be canceled by considering the sum of (1) and (4) and the difference between (2) and (3). Thus we obtain the initial state

$$|i\rangle = \sum_{s\sigma}\left[\alpha_{s\sigma}\left(a^\dagger_{1\sigma}a^\dagger_{2s} - a^\dagger_{2\sigma}a^\dagger_{1s}\right) + \beta_{s\sigma}\left(a^\dagger_{1\sigma}a^\dagger_{1s} + a^\dagger_{2\sigma}a^\dagger_{2s}\right)\right]|0\rangle , \quad (5)$$

which leads to a final state of the required form

$$|f\rangle = \sum_{s\sigma}\left[\alpha_{s\sigma}\left(a^\dagger_{4\sigma}a^\dagger_{3s} - a^\dagger_{3\sigma}a^\dagger_{4s}\right) + i\beta_{s\sigma}\left(a^\dagger_{4\sigma}a^\dagger_{3s} + a^\dagger_{3\sigma}a^\dagger_{4s}\right)\right]|0\rangle .$$

   (a) For two photons with the same polarization $s = \sigma$, and using the commutativity of the creation operators of bosons one gets the initial state

$$|i\rangle = \frac{1}{2}\left[\left(a^\dagger_{1x}\right)^2 + \left(a^\dagger_{2x}\right)^2\right]|0\rangle .$$

   (b) For general polarizations s and σ the initial state is (5).

2. (a) For electrons with identical spin $s = \sigma$. Substituting this in (5) and using the anti-commutativity of fermionic creation operators the initial state is

$$|i\rangle = a^\dagger_{1\uparrow}a^\dagger_{2\uparrow}|0\rangle .$$

   (b) For electrons with non-identical spins the required initial state is simply (5).

3. Non-identical particles have distinct creation operators $a^\dagger_{i\sigma}$ and $b^\dagger_{js}$. By using the same arguments as used in question 1, it follows that the general initial state satisfying the requirements is

$$|i\rangle = \sum_{s\sigma}\left[\alpha_{s\sigma}\left(a^\dagger_{1\sigma}b^\dagger_{2s} - a^\dagger_{2\sigma}b^\dagger_{1s}\right) + \beta_{s\sigma}\left(a^\dagger_{1\sigma}b^\dagger_{1s} + a^\dagger_{2\sigma}b^\dagger_{2s}\right)\right]|0\rangle . \quad (6)$$

   The final state obtained from (6) is

$$|f\rangle = \sum_{s\sigma}\left[\alpha_{s\sigma}\left(a^\dagger_{4\sigma}b^\dagger_{3s} - a^\dagger_{3\sigma}b^\dagger_{4s}\right) + i\beta_{s\sigma}\left(a^\dagger_{4\sigma}b^\dagger_{3s} + a^\dagger_{3\sigma}b^\dagger_{4s}\right)\right]|0\rangle .$$

(a) Since the particles are non-identical, the spins/polarizations have no influence and the initial state is (6).

(b) The same as above. The initial state is (6).

4. The collapse is caused by groups of states in the wave-function that interfere destructively. Thus, it can be estimated using the phase difference along the width of the wave-function:

$$t_c \sim \frac{1}{\Omega_{n=1100} - \Omega_{n=1000}} = \frac{1}{(4g^2 \cdot 1100)^{1/2} - (4g^2 \cdot 1000)^{1/2}} = \frac{1}{20(\sqrt{11} - \sqrt{10})g} .$$

5. Revival stems from all the oscillation modes being in the same phase. Hence the time depends only on the phase difference between near modes and the result is identical to the one of a coherent state:

$$t_r = \frac{2\pi}{\Omega_{\langle n\rangle+1} - \Omega_{\langle n\rangle}} \approx \frac{2\pi\langle n\rangle^{1/2}}{g} .$$

6. This potential is harmonic potentials in the x- and y-axes, so the energy levels are $E_{n_x,n_y} = (n_x + n_y + 1)\hbar\omega$. The degeneracy of the n^{th} level is $g_n = 2(n+1)$ (the factor of two is from the two-fold degeneracy of the fermion spin).
The total no. of states is $N_s = \sum_{m=0}^{n} g_n = (n+2)(n+1)$. Taking into account even or odd number of particles N and the fact that $n = \frac{\epsilon_F}{\hbar\omega} - 1$ we get the *exact* equation for ϵ_F:

$$\left(\frac{\epsilon_F}{\hbar\omega} - 1\right)\frac{\epsilon_F}{\hbar\omega} < N \leq \left(\frac{\epsilon_F}{\hbar\omega} + 1\right)\frac{\epsilon_F}{\hbar\omega} .$$

7. Using the Thomas–Fermi approximation $k_F^2 = \frac{2m(\epsilon_F - V(r))}{\hbar^2}$ the particle density in a small neighborhood at a distance r from the origin is

$$\rho = 2\frac{\pi k_F^2(r)}{(2\pi)^2} = \frac{2m(\epsilon_F - \frac{1}{2}m\omega^2 r^2)}{2\pi\hbar^2} .$$

The particles have enough energy to reach a radius of $r_{\max}$ satisfying $\epsilon_F = \frac{1}{2}m\omega^2 r_{\max}^2$. The total number of particles is

$$N = \int_0^{r_{\max}} 2\pi r\rho dr = \frac{\epsilon_F^2}{(\hbar\omega)^2} .$$

and therefore, $\epsilon_F = \sqrt{N}\hbar\omega$.

8. In the three-dimensional case

$$\rho = 2\frac{4\pi k_F^3(r)/3}{(2\pi)^3} = \frac{[2m(\epsilon_F - V(r)]^{3/2}}{3\pi^2\hbar^3} .$$

All the states up to $\epsilon = \epsilon_F$ are occupied, so the total number of particles is

$$N = \int_0^{r_{\max}} 4\pi r^2 \rho dr = \frac{4(2m)^{3/2}}{3\pi\hbar^3}\epsilon_F^{3/2}\left(\frac{2\epsilon_F}{m\omega^2}\right)^{3/2}\int_0^1 (1-u^2)^{3/2}u^2 du = \frac{\epsilon_F^3}{3(\hbar\omega)^3} ,$$

where $u = r/r_{\max}$ has been substituted and the given definite integral was used. Finally, $\epsilon_F = (3N)^{1/3}\hbar\omega$.