Solution for the Quantum Physics 1 Exam from Feb. 19, 2004

1. the fermion kinetic term yields

$$\sum_{\mathbf{p},\mathbf{q}} \langle \mathbf{q}| \frac{\hbar^2 \mathbf{p}^2}{2m} |\mathbf{p}\rangle a^\dagger_{\mathbf{p}} a_{\mathbf{p}} = \sum_{\mathbf{p}} \frac{\hbar^2 \mathbf{p}^2}{2m} a^\dagger_{\mathbf{p}} a_{\mathbf{p}} \ .$$

The $f(\mathbf{x})$ term is obtained using ψ-operators:

$$\begin{aligned} \hat{f} &= \int d^3x \hat{\psi}^\dagger(\mathbf{x}) f(\mathbf{x}) \hat{\psi}(\mathbf{x}) = \sum_{\mathbf{p},\mathbf{q}} a^\dagger_{\mathbf{q}} a_{\mathbf{p}} \int d^3x f(\mathbf{x}) e^{-i(\mathbf{q}-\mathbf{p})\cdot\mathbf{x}} = \\ &= \sum_{\mathbf{p}} \left(b_{\mathbf{q}_0} a^\dagger_{\mathbf{p}+\mathbf{q}_0} a_{\mathbf{p}} + b^\dagger_{\mathbf{q}_0} a^\dagger_{\mathbf{p}-\mathbf{q}_0} a_{\mathbf{p}} \right) \ . \end{aligned}$$

Thus the Hamiltonian is

$$H = \sum_{\mathbf{p}} \frac{\hbar^2 \mathbf{p}^2}{2m} a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + \lambda \sum_{\mathbf{p}} \left(b_{\mathbf{q}_0} a^\dagger_{\mathbf{p}+\mathbf{q}_0} a_{\mathbf{p}} + b^\dagger_{\mathbf{q}_0} a^\dagger_{\mathbf{p}-\mathbf{q}_0} a_{\mathbf{p}} \right) \ .$$

2. Restricting the momentum of the fermions to the values $\mathbf{p} = 0, \mathbf{q}_0$ and adding the kinetic term for the bosons we have

$$H = \frac{\hbar^2 \mathbf{q}_0^2}{2m} a^\dagger_{\mathbf{q}_0} a_{\mathbf{q}_0} + \lambda \left(b_{\mathbf{q}_0} a^\dagger_{\mathbf{q}_0} a_0 + b^\dagger_{\mathbf{q}_0} a^\dagger_0 a_{\mathbf{q}_0} \right) + \frac{\hbar^2 \mathbf{q}_0^2}{2m} b^\dagger_{\mathbf{q}_0} b_{\mathbf{q}_0} \ ,$$

which is readily identified as the Jaynes–Cumming Hamiltonian. Since the amplitude for being in the state $|\mathbf{q}_0, n = 0\rangle$ changes as $\cos(\frac{\Omega_0 t}{2})$, where $\Omega_0 = \frac{2\lambda}{\hbar}$, we obtain the state $|0, n = 1\rangle$ when $t = \frac{\pi\hbar}{2\lambda}$.

3. The kinetic correction is $E_0^{(1)} = -\frac{\langle \mathbf{p}^4 \rangle}{8m^3c^2}$. Since $p_x = i\sqrt{\frac{m\hbar\omega}{2}}(a^\dagger_x - a_x)$,

$$p_x^2 = -\frac{m\hbar\omega}{2} \left[(a^\dagger_x)^2 - a^\dagger_x a_x - a_x a^\dagger_x + a_x^2 \right] \ ,$$

so

$$\begin{aligned} \langle p_x^4 \rangle &= \frac{m^2\hbar^2\omega^2}{4} \langle (a^\dagger_x)^4 + (a^\dagger_x a_x)^2 + (a_x a^\dagger_x)^2 + a_x^4 - (a^\dagger_x)^3 a_x - (a^\dagger_x)^2 a_x a^\dagger_x + (a^\dagger_x)^2 a_x^2 - a^\dagger_x a_x (a^\dagger_x)^2 \\ &\quad + a^\dagger_x a_x^2 a^\dagger_x - a^\dagger_x a_x^3 - a_x (a^\dagger_x)^3 + a_x (a^\dagger_x)^2 a_x - a_x a^\dagger_x a_x^2 + a_x^2 (a^\dagger_x)^2 - a_x^2 a^\dagger_x a_x + a_x^3 a^\dagger_x \rangle \\ &= \frac{3m^2\hbar^2\omega^2}{4} \ , \end{aligned}$$

and

$$\langle p_x^2 \rangle = -\frac{m\hbar\omega}{2} \langle (a^\dagger_x)^2 - a^\dagger_x a_x - a_x a^\dagger_x + a_x^2 \rangle = \frac{m\hbar\omega}{2} \ .$$

Obviously, these results apply to the y and z axes as well since the ground state is $|n_x = 0, n_y = 0, n_z = 0\rangle$. Hence,

$$\begin{aligned} \langle \mathbf{p}^4 \rangle &= \langle \left(p_x^2 + p_y^2 + p_z^2 \right)^2 \rangle = \langle p_x^4 + p_y^4 + p_z^4 + 2p_x^2 p_y^2 + 2p_x^2 p_z^2 + 2p_y^2 p_z^2 \rangle = \\ &= 3\frac{3m^2\hbar^2\omega^2}{4} + 6 \left(\frac{m\hbar\omega}{2} \right)^2 = \frac{15m^2\hbar^2\omega}{4} \end{aligned}$$

and

$$E_0^{(1)} = -\frac{15\hbar^2\omega^2}{32mc^2} \ .$$

4. The field dependent corrections are

$$E_0^{(2)} = \langle -\frac{e\hbar}{2m^2c^2r}\frac{d\Phi}{dr}\mathbf{L}\cdot\mathbf{S} - \frac{e\hbar^2}{8m^2c^2}\nabla\cdot\mathbf{E}\rangle \ .$$

The ground state has no orbital angular momentum so $\langle\mathbf{L}\cdot\mathbf{S}\rangle = 0$. Using

$$e\nabla\cdot\mathbf{E} = -e\nabla^2\Phi = 3m\omega^2 \ ,$$

the second correction is

$$E_0^{(2)} = -\frac{3\hbar^2\omega^2}{8mc^2} \ .$$

5. In the drawn beam-splitter the creation operators transform as

$$a_1^\dagger = \frac{1}{\sqrt{2}}\left(a_4^\dagger + ia_3^\dagger\right) \ , \quad a_2^\dagger = \frac{1}{\sqrt{2}}\left(a_3^\dagger + ia_4^\dagger\right) \ ,$$

from which the transformation of the variables p and x is obtained

$$\begin{aligned} x_3 &= \frac{1}{\sqrt{2}}(x_2 - p_1) \ , \\ x_4 &= \frac{1}{\sqrt{2}}(x_1 - p_2) \ , \\ p_3 &= \frac{1}{\sqrt{2}}(x_1 + p_2) \ , \\ p_4 &= \frac{1}{\sqrt{2}}(x_2 + p_1) \ . \end{aligned}$$

The number operator can be written as $n_4 = \frac{1}{2}\left(p_4^2 + x_4^2 - 1\right)$ and

$$\langle p_4^2\rangle = \frac{1}{2}\langle(x_2+p_1)^2\rangle = \frac{1}{2}\left(\langle x_2^2\rangle + 2\langle x_2\rangle\langle p_1\rangle + \langle p_1^2\rangle\right) \ .$$

$\langle p_1\rangle$ and $\langle p_2\rangle$ are indetermined because the uncertainty of p_1 and p_2 is infinite for the state $|x=1\rangle$ and $\langle p_1^2\rangle$ diverges faster than $\langle p_1\rangle$ so $\langle n_4\rangle$ is infinite.
Similarly,

$$\langle x_4\rangle = \frac{1}{\sqrt{2}}\left(\langle x_1\rangle - \langle p_2\rangle\right)$$

is indetermined and

$$\langle x_4^2\rangle = \frac{1}{2}\langle(x_1-p_2)^2\rangle = \frac{1}{2}\left(\langle x_1^2\rangle - 2\langle x_1\rangle\langle p_2\rangle + \langle p_2^2\rangle\right)$$

is infinite (it diverges faster than $\langle x_4\rangle$) so Δx_4^2 diverges.

6. Here $\langle p_4^2\rangle$ is infinite because $\langle p_1^2\rangle$ is infinite so $\langle n_4\rangle$ is infinite. For a coherent state with $\alpha = 1$ $\langle p_2^2\rangle = \frac{1}{2}$ and $\langle p_2\rangle = 0$ so

$$\begin{aligned}
\langle x_4\rangle &= \frac{1}{\sqrt{2}}\left(\langle x_1\rangle - \langle p_2\rangle\right) = \frac{1}{\sqrt{2}}(1-0) = \frac{1}{\sqrt{2}} \ , \\
\langle x_4^2\rangle &= \frac{1}{2}\left(\langle x_1^2\rangle - 2\langle x_1\rangle\langle p_2\rangle + \langle p_2^2\rangle\right) = \frac{1}{2}\left(1 - 0 + \frac{1}{2}\right) = \frac{3}{4} \ , \\
\Delta x_4^2 &= \langle x_4^2\rangle - \langle x_4\rangle^2 = \frac{1}{4} \ .
\end{aligned}$$

7. The final states are the coherent states $|\alpha_4 = \frac{1+i}{\sqrt{2}}\rangle$ so

$$\begin{aligned}
\langle n_4\rangle &= \alpha_4^*\alpha_4 = 1 \ , \\
\langle x_4\rangle &= \sqrt{2}\mathrm{Re}\alpha_4 = 1 \ , \\
\Delta x_4^2 &= \frac{1}{2} \ .
\end{aligned}$$