\[ H \Psi(1, 2, \ldots, A) = E \Psi(1, 2, 3, \ldots, A) \]

\[ H = \sum_{i=1}^{A} -\frac{k^2}{2m} \nabla_i^2 + \sum_{ij} V_{ij} \]

\text{two-body } N-N \text{ interaction}

Direct sources of information about \( V_{ij} \):

1. The deutron
2. N-N scattering.

* The two-body interaction must be a scalar in spin-space and for the nuclear part it must be an isoscalar.
The Deuteron and the Two-Body Nuclear Force.

The deutron (an n-p system) is the only bound two-nucleon system. It provides information about the N-N force (in addition to N-N scattering).

\[
\text{from } M_d = 1876.1244 \text{ MeV} \\
M_n c^2 + M_p c^2 = M_d c^2 = (B.E)_d \\
939.57 \text{ MeV} + 938.78 - 1876.12 = 2.22 \text{ MeV}
\]

(Also from the reaction \( n+p \to d+t \))

The deuteron has no excited bound state.

Other properties of the D:

- Binding Energy, BE: 2.224 MeV
- Spin and parity, J^T: 1^+
- Isospin, T: 0
- Magnetic dipole moment, \( \mu_d \): 0.857 MN
- Electric Quadrupole moment, \( Q_d \): 0.286 e fm^2
- Radius, \( R_d \): 1.963 fm

\text{Note: } 1.1 A^{1/3} (1.1 \times 1.4 = 1.55 \text{ fm})
Downon Cont'd.

Parkan \( \pi = +1 \Rightarrow \) even \( L \), \((-1)^L = \pm 1\)

\( J = 1^+ \) is obtained from \( L = 0, S = 2 \) \((^2S)\)

but this state is higher in energy (Higher \( L \)'s cannot give \( J = 1^+ \))

We will see that there is a small admixture of the \((L = 2, S = 1)\) (D-state) into the \((L = 0, S = 1)\). \( \text{R}^3\text{A} \)

Isospin \( T_\alpha = 0 \). Since \( L = 0, S = 1 \)

is a symmetric state, their function must be anti-symmetric, thus \( T = 0 \)

So: \((T = 0, T_\alpha = 0)\).

\[ \text{Magnetic dipole moment.} \]

\[ \text{Orbital magnetic dipole moment; spin} \]

\[ \begin{align*}
M_p &= \frac{e}{2\hbar} l_p \quad \text{orbital} \quad l_p \quad \text{orbital angular momentum} \\
M_m &= 0 \\
M_s &= g_s(\cdot)^\frac{\hbar}{2} \\
 M_i &= \frac{g_i}{2} \quad \text{isospin} \quad g_i = \frac{2}{3} \quad g_s = \frac{1}{2} \\
 g_s(\cdot) &= g_p - 2 \mu_p = 5.58 \quad - \text{proton} \\
g_i &= 2 \mu_n = -3.82 \quad - \text{neutron} \\
 \end{align*} \]

Operators: \( \mu_i = g_p \vec{S}_p + g_n \vec{S}_n + \ell_p \)

\( \mu_\ell = \frac{1}{2} \ell \)
Definition of Magnetic Dipole Moment

\[ \mathbf{M}_d \equiv \langle J, M = J | \mathbf{M}, z | J, M = J \rangle \]

component of the vector

Landaú formula

\[ \langle J M | M_z | J M \rangle = \frac{M}{J(J+1)} \langle J M | \mathbf{M} \cdot \mathbf{J} | J M \rangle \]

Follows from the Wigner-Eckart theorem

\[ \mathbf{M}_d = \frac{1}{2} (\mathbf{g}_p + \mathbf{g}_n) \mathbf{S} + \mathbf{g}_p - \mathbf{g}_n) \mathbf{S}_d + \mathbf{L} \]

For an expectation value \( \langle \mathbf{g}_p - \mathbf{g}_n \rangle = 0 \)

\[ \langle J M | M_z | J M \rangle = \frac{M}{J(J+1)} \langle J M | \{ (\mathbf{g}_p + \mathbf{g}_n) \mathbf{S} \cdot \mathbf{J} \} + (\mathbf{L} \cdot \mathbf{J}) \rangle | J M \rangle \]

\[ \mathbf{S} \cdot \mathbf{J} = \mathbf{S} (\mathbf{L} + \mathbf{S}) = \mathbf{S} \cdot \mathbf{L} + \frac{1}{2} (J - L^2 - S^2) \]

\[ \mathbf{L} \cdot \mathbf{S} = L^2 + \frac{1}{2} (J^2 + L^2 - S^2) \]

\[ \mathbf{M}_d = \frac{1}{4(J+1)} \left\{ (\mathbf{g}_p + \mathbf{g}_n)(J(J+1) - L(L+1)) \right. \]

\[ \left. + S(S+1) \right) \right\} \]}

For the \( 3S \), \( (L=0, S=J=1) \)

\[ \mathbf{M}_d = \frac{1}{3} \{ (\mathbf{g}_p + \mathbf{g}_n)(2+2) + (2-2) \} \]

\[ \mathbf{M}_d = \frac{4}{3} (\mathbf{g}_p + \mathbf{g}_n) \mathbf{L} = \mathbf{L} \mathbf{M}_d + \mathbf{L} \mathbf{M}_d = | \mathbf{M}_d + \mathbf{M}_d | \]
\[ M_d(3_s) = M_p + M_n = 0.879\,9805\,\mu N \]
\[ M_d - M_d(3S) = (0.8574\,838 - 0.879\,9805)\,\mu N \]
\[ = -0.022\,367\,\mu N \]

Note that from the expression for \( M_d \) for the \(^3\)D, one obtains:
\[ M_d(3\,D_1) = \frac{1}{2}\left(3p^2 + 9n\right)(-2) + 6^2 = 0.810\,\mu N \]

Admixture of the \(^3\)D state

\[ |4D\rangle = \alpha |3\,S_1\rangle + \beta |3\,D_1\rangle \]

\[ \alpha^2 + \beta^2 = 1. \]

\[ M_d = \alpha^2 M_d(3\,S_1) + \beta^2 M_d(3\,D_1) \]

\[ = 0.8574\,\mu N \]

Experimental

From here one finds:
\[ \beta^2 = 0.04 \quad \text{is} \quad 4^\circ, \text{admixture.} \]
Take \( H_d \) to be the Hamiltonian for the deuteron.

\[
\left( H_d = \frac{\hbar^2}{2m}\nabla^2 + V \right)
\]

\( H_d |\psi\rangle = \epsilon_d |\psi\rangle \)

Take as the basis two states:

\[ T \text{ : } 1^1S, D \]

and diagonalize the Hamiltonian

\[
\begin{pmatrix}
1 & \langle \psi | H | 1^1S \rangle & \langle \psi | H | 1^3D \rangle \\
\langle \psi | H | 1^1S \rangle & \langle \psi | H | 1^1D \rangle \\
\langle \psi | H | 1^3D \rangle & \langle \psi | H | 1^3D \rangle
\end{pmatrix}
\]

The result will be:

\[ |\psi\rangle = \alpha |1^1S\rangle + \beta |1^3D\rangle \quad \text{a.s.} \]

and an excited state

\[ (\Psi) = -\beta |1^1S\rangle + \alpha |1^3D\rangle \]

\[ \Psi (1^3D) \to 0. \]
Deuteron Quadrupole Moment

In quantum mechanics in the multipole expansion of charge distribution only even electric moments are allowed. Apart from the monopole ($L=0$) the lowest nonvanishing moment is the quadrupole ($L=2$).

The measured quadrupole moment for the deuteron is: $Q_{0} = 0.28590 \, \text{cm}^2$.

For a spherical distribution:
\[ \frac{1}{3} \langle r^2 \rangle = \langle x^2 \rangle = \langle y^2 \rangle = \langle z^2 \rangle \]
\[ \langle r^2 \rangle = 3 \langle z^2 \rangle \]

For a deformed nucleus the quadrupole operator $Q_0 = \epsilon (3z^2 - r^2) \neq 0$

\[ (Q_0 > 0 \text{ prolate}) \]
\[ (Q_0 < 0 \text{ oblate}) \]

\[ Q_0 = \epsilon (z^2 - r^2) = \epsilon r^2 (3 \cos^2 \theta - 1) = \sqrt{\frac{16\pi}{5}} Y_0^2 (0, 0) \epsilon r^2 \text{ spherical harmonics.} \]
The quadrupole moment is defined as

\[ Q = \langle J M = J | \hat{Q}_0 | J M = J \rangle \]

\[ = \langle J M = J | Y_2^0 | J M = J \rangle \sqrt{\frac{15 \pi}{2}} \]

\((J, z, J)\) must obey the triangle relation.

If \( J = 0 \), \( Q = 0 \); also for \( l = 0 \).

Therefore in the deutron for a "pure" \( ^3S_1 \) ground state \( Q = 0 \). So, if \( Q \neq 0 \) it means that there must be an admixture of the \( ^3D \), component in the deutron.

Taking \( |4d\rangle = \alpha |^3S_1\rangle + \beta |^3D_1\rangle \)

\[ \langle 4d | \hat{Q}_0 | 4d \rangle = \alpha^2 \langle ^3S_1 | \hat{Q}_0 | ^3S_1 \rangle + \]

\[ 2 \alpha \beta \langle ^3S_1 | \hat{Q}_0 | ^3D_1 \rangle + \beta^2 \langle ^3D_1 | \hat{Q}_0 | ^3D_1 \rangle \]

The first term (with \( \alpha^2 \)) is zero.

The last term is small because \( \beta^2 \)

is small (\( \beta \approx 0.04 \)) so the \( 2 \alpha \beta \) is the important one.
Appendix A

Irreducible Spherical Tensors

\[ P(x, y, z) \]

\[ P(x', y', z') \]

Rotation \( R \) in space

\[ Y^p_{x'}(\theta', \phi') = \sum Y^k_x(0, \theta) D_{x'x}(R) \]

\[ \frac{\mathbf{r}_1}{\mathbf{r}} = R \frac{\mathbf{r}_1}{\mathbf{r}} \]

\[ \frac{\mathbf{r}_1}{\mathbf{r}} = R^{-1} \frac{\mathbf{r}_1}{\mathbf{r}} \]

\( \theta', \phi' \) refer to the same physical point \( P \) in the new (primed) frame of reference. There are \( 2(k+1) \)

\[ \{x = (x, -x, 1), \ldots, x, k \} \]

\( Y^k_x \) components.

The set is irreducible, that is there is no subset that transforms into itself under a rotation.

These functions are the spherical harmonics that obey

\[ L_x Y^p_x = k(k+1) Y^p_x \]

\[ L_{-2} Y^p_x = \Delta \phi Y^p_x \]
and also
\[(L_x \pm i L_y) Y_{\ell}^\mu = \sqrt{\kappa(n+1) - \kappa(\ell \pm 1)} \ Y_{\ell \pm 1}^\mu\]

**Generalization.**

An irreducible tensor of degree \( k \) is:
\[\left(T^{x'}_{x'}\right)'(R) = \sum_x T^x_x(\kappa) D_{xx'}(R)\]

Irreducible tensor field:
\[\left(T^{x'}_{x'}\right)'(\tilde{\tau}^i) = \sum_x T^x_x(\kappa) D_{xx'}(R)\]

In the primed system.

That is, the transformation law of a tensor of rank \( k \) is the same as of \( T^x_x \).

\[J_z T^x_x = x T^x_x\]

\[(J_x \pm J_y) T^x_x = \sqrt{\kappa(n+1) - \kappa(\ell \pm 1)} \cdot T^x_{x \pm 1}\]

If \( T^x_x \) operators, then
\[\left[J_z, T^x_x\right] = x T^x_x\]

\[\left[J_x \pm J_y, T^x_x\right] = \sqrt{\kappa(n+1) - \kappa(\ell \pm 1)} \cdot T^x_{x \pm 1}\]

\[\frac{\hat{A}_0}{\tau_1} = R^2\]

\( \tau_1 \) refers to the same physical point \( P \) but in the rotated frame of reference.

**Example:** E-M transition:

\[\langle J_{\mathbf{1}} | d_2 | J_{\mathbf{1}}' \rangle\]

Dipole transition between two states

\[d_2 = e \mathbf{z} = e \cos \theta\]

\[\langle J_{\mathbf{1}} | e r Y_{0}(\theta) | J_{\mathbf{1}}' \rangle \neq 0\]

Only if \( J' = J+1, J, J-1 \)

\( (\Leftrightarrow J = 0 \Rightarrow J' = J = 1 \text{ only}) \).

Note however, that irreducible tensors are more general, they may be formed from spin variables and may have half-integer ranks.
Example: Any tensor of second degree \([T_{ik}]\) can be decomposed into three irreducible tensors:

\[T_{ik} = S_{ik} + A_{ik} + T_{ik}\]

\[T_{ik} = \frac{1}{3} \left[ T_{1i} + T_{2i} + T_{3i} \right] \Rightarrow \frac{1}{3} \text{trace } T\]

Tensor of degree 0 (scalar) \([k=0]\)

\[A_{ik} = \frac{1}{2} \left( T_{ik} - T_{ki} \right) \Rightarrow \text{antisymmetric tensor has only 3 components (thus it is an vector \((\text{axial})\))}\]

\[S_{ik} = \frac{1}{2} \left( T_{ik} + T_{ki} \right) - T_{ik} \Rightarrow \text{symmetric tensor has 5 independent components \([k=2]\)}\]

\[1 + 3 + 5 = 9\]

these are three irreducible tensors.

Note that if you take an \(l=1\) (p-state) particle and couple it to another particle in a \(p\) state \((l=1)\) you obtain the following: For \(l=1\) the \(s.f.\) is \(xf(r)\) \(yf(r)\) \(zf(r)\) \(rf(r)\) \(rt(r)\)

Now when you take \(l \times l = l^2\) you obtain functions \(xix'(r) f(r)\) \(yiy'(r) f(r)\) \(ziz'(r) f(r)\) \(xyf(r)\) \(yzf(r)\) \(ztf(r)\) \(xrf(r)\) \(yr'f(r)\) \(zt'f(r)\)

We can reduce this to \([k=0, 1, 2]\). When we couple \(l=1\) to \(l=1\) we get \(L=0, 1, 2\), which are described by \(2L+1\) components \((1, 3, 5)\)
Adding spin (angular momenta, total spins).

Given: \( j_1, j_2 \) represented by w.f.
\[ \Psi(j_1 m_1, j_2 m_2) \]
\[ m_1 = -j_1, \ldots, j_1, \quad m_2 = -j_2, \ldots, j_2 \]

The wave function of total angular momentum \( J \) is:
\[ \Psi(j_1, j_2, J M) = \sum_{m_1, m_2} \left( j_1 m_1, j_2 m_2 | J M \right) \Psi(j_1 m_1) \Psi(j_2 m_2) \rightarrow \text{C.G. coeff.} \]
\[ M = m_1 + m_2 \]
\[ J = j_1 + j_2, j_1 + j_2 - 1, \ldots, |j_1 - j_2| \]

The same for spherical tensors:
\[ T^k_x (k_1, k_2) = \sum_{l \geq J, \ell \geq j_1} \sum_{\kappa} \left( k_1 \ell x_1, k_2 \ell x_2 | k \kappa \right) T^{k_1}_{x_1} T^{k_2}_{x_2} \]
\[ x_1 + x_2 = x \quad \text{and} \quad k = k_1 + k_2, k_1 + k_2 - 1, \ldots, |k_1 - k_2| \]

So these tensors are like \( \Psi \) functions, in fact \( T^k_x \) are spherical tensors. Therefore if we have
\[ \langle JM | T^k_{x_1} | J'M' \rangle \]
this matrix element is non-zero only when \( M + x = M' \)
and when \( J_1 = J + k \), or \( J + k - 1, \ldots, |J - k| \)
(the same as saying \( J + k \geq J' \geq |J - k| \))
Appendix I

The Wigner-Eckart Theorem

Let $T_x^k$ be an irreducible spherical tensor.

Then

$$\langle j' m' | T_x^k | j m \rangle =$$

$$= (-1)^{j'-m} \left( \frac{j' k' {j'}^*}{m' \times m'!*} \right) \langle j' \| T_x^k \| j \rangle$$

(reduced matrix element)

$$\left( \frac{j_1, j_2, j_3}{m_1, m_2, m_3} \right) = (-1)^{j_1-j_2-m_3} \frac{1}{\sqrt{2j_3+1}} \left( j_1, j_2, m_1, m_2, 1, j_3, -m_3 \right)$$

(Clebsch-Gordan)

$$\left( \frac{j_1, j_2, j_3}{m_1, m_2, m_3} \right) = 0 \text{ if } m_1+m_2+m_3 \neq 0$$